As we review Paul's letters to us and his comments at B.U., it seems that he thought we were trying to revive his old nemesis that involved the fallacy of large numbers.[1] The key difference between the two approaches is whether the investor is betting more in total or just spreading out his bets over more opportunities (while holding the total exposure constant). The latter reduces risk while the former is Samuelson's fallacy.

On its face (and perhaps in the abstract), it might seem like our lifecycle investing strategy is designed to help the investor simply get more years of exposure to the market as in "If you have a longer run, then it's even safer to invest in stocks." But that is not our strategy. In our risk-reducing implementation, we want people to borrow to invest more when young and then invest less when older. The lifetime exposure to stocks is held constant. Compare the following two investment paths:

Option 1

Year 1	Invest $1
Year 2	Invest $2
Year 3	Invest $3

Option 2

Year 1	Invest $2
Year 2	Invest $2
Year 3	Invest $2

Our view is that option 2 is the safer bet. The total dollar amount invested is six dollar years in both cases, but it is more evenly spread out in the second case. Both strategies have the same expected return, but the second has less risk. The variance of option 1 is $14\sigma^2$, while the variance of option 2 is $12\sigma^2$.

So why doesn't everyone follow option 2? The reason is that most people don't have $2 to invest in year 1. The only way to invest more when young is to use leverage. Here the point of leverage is to do a better job spreading out the risk across time. If you don't use leverage, then you are giving up the chance to diversify your investments across the first two or three decades of your working life.

We also propose an implementation of the lifecycle strategy which holds risk constant and increases lifetime exposure to equities. The point here is to come closer to the utility-maximizing allocation. In Samuelson's 1969 article, his investor had the good fortune of being born with all of his wealth upfront. For such an investor (and given constant relative risk aversion), this

[1] The theory Samuelson disproved is this: If you can invest long enough, any risky bet with a positive expectation will become attractive. The law of large numbers doesn't save you here as the gamble is adding up more bets, not averaging across them. The proof is remarkably simple. Say you wouldn't take a gamble, no matter your income level, if offered only once. Then you wouldn't take this gamble multiple times either. Here's why. If you have the chance to decline the last bet, you would, since you don't like this gamble at any income. Better still would be to also decline the penultimate bet. This series takes you all the way back to the first gamble. That said, it is true that if you can spread your money across enough (independent) bets then any gamble with a positive expectation will become attractive. Instead of betting $100 once, you'd rather bet $1 one hundred times or 1¢ ten thousand times. By splitting your bet into smaller portions, the final result converges to the expected value.

investor should allocate his money between stocks and bonds in the same fashion regardless of age.

For purposes of illustration, say that this optimal allocation was 50% stocks and 50% bonds. The question is what do the rest of us do since we don't have all of our lifetime wealth given to us upfront. Say that your lifetime wealth (W) is $1 million and so you'd like to have 50% or $500,000 in stocks. The problem is that you only have $50,000 in cash. The rest of your wealth is tied up in your human capital and will be transformed into financial capital slowly over time. The Samuelson dictum does not say to invest half of the $50,000 in equities. Even if you put all of the $50,000 into stocks, you will still be holding a $950,000 bond in the form of your human capital. (Here we are making the critical assumption that your human capital is more like a bond than a stock.) Our strategy is to leverage the $50,000 at 2:1 in order to invest $100,000 in equities.

The key point here is that we are not borrowing to invest 2W in stocks. Rather, we are borrowing or employing leverage in order to invest W/2 in stocks. If the investor had access to W/2 today, there would be no need to employ leverage. But just as most investors do not have the capital to buy a house without leverage, they don't have the capital to buy the desired allocation of stocks without leverage.[2] And while it may look like borrowing, it is really selling some of your human capital bond portfolio.

We wish we had been able to put it this clearly to Paul—including pointing out to him that his 1969 article itself said a high future salary "does justify leveraged investment financed by borrowing against future savings."[3] While we have sadly lost that opportunity, we can do our best to help others see the practical implication of his beautiful result. If nothing else, it shows how profound his work was. If this were all obvious, then we wouldn't have had to provide this explanation.

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[2] Of course, once you borrow to buy stocks then the interest you pay becomes the relevant bond rate. In a continuous time model,

$$\text{Samuelson Share} = \text{Equity Premium} / (\sigma^2 * \text{RRA}).$$

A higher interest rate implies a lower equity premium and thus a lower fraction invested in stocks. What we show in our academic article and in our book is that it is possible to borrow at rates very close to the bond rate in order to buy stocks with up to 2:1 leverage. Once the leverage gets to be much higher than 2:1, the incremental cost of borrowing becomes significantly larger. That is why someone with $50,000 doesn't invest at 10:1 leverage to get to the 50% Samuelson Share. At the interest rate that corresponds to borrowing at 10:1 leverage, the Samuelson Share would be much lower than 50%—it would have likely dropped to 0%.

[3] Samuelson (1969) in contrasting a "businessman," who has future income, with a "widow," who does not, said:

> [The businessman] can look forward to a high salary in the future; and with so high a present discounted value of wealth, it is only prudent for him to put more into common stocks compared to his present tangible wealth, *borrowing if necessary for the purpose* (italics added), or accomplishing the same thing by selecting volatile stocks that widows shun. . . . [A high salary in the future] does justify leveraged investment financed by borrowing against future earnings. But it does not really involve any increase in relative risk-taking once we have related what is at risk to the proper larger base. (Admittedly, if market imperfections make loans difficult or costly, recourse to volatile, "leveraged" securities may be a rational procedure.)

Appendix:

In Paul's letter to us, he used the example of Dan Bernoulli as an investor. Dan has a log utility function and faces two investment options:

Option A is a riskless bond that pays 0% interest

Option B is a stock that returns either 4x or x/4, each with equal likelihood.

As Paul noted in his letter, the investor would ideally allocate 50% of his wealth to Option B. This is true whether his investment horizon is one period or one hundred periods.

Consider the case where our investor has wealth W but has access to only some fraction of that amount today, say fW. If $f < 25\%$ and the investor can borrow at 0%, then he is better off employing 2:1 leverage than no leverage. Below we show the gain in expected utility.

How much better off is our investor? Consider the case where the investor has W = $1 million but he has access to only $100k ($f = 0.10$). If the fraction of $100k invested is 50%, then Dan has exposed 5% of his wealth to equities and expected utility is

$$EU = \ln(W) + (1/2)\,[\ln(4*(0.50f)+(1-0.50f))] + (1/2)\,[\ln((0.50f)/4 + (1-0.50f))]$$

$$= \ln(W) + (1/2)\,[\ln(4*0.05+0.95) + \ln(0.05/4 + 0.95)]$$

$$= \ln(W) + (1/2)\ln(1.15*0.9625)$$

$$= \ln(W) + 0.0508$$

Since ln(W) is a constant, W is often ignored. In our case, it matters as, f, the fraction of W which is available today is dependent on W. If Dan could put W/2 in option B he would. But he doesn't have access to the capital today. The roundabout way of gaining more access to that locked-up human capital is to employ leverage with the capital that he does have access to.

If the tangible fraction of his wealth, $100k, is invested at 200%, then Dan has exposed 20% of his wealth to equities and so expected utility is

$$EU = \ln(W) + (1/2)\,[\ln(4*0.20+0.80) + \ln(0.20/4 + 0.80)]$$

$$= \ln(W) + (1/2)\ln(1.6*0.85)$$

$$= \ln(W) + 0.1537$$

This is a 203% improvement (0.1537/0.0508 – 1) in the expected utility that comes from the investment activity. And even this small amount of leverage provides 69% of the potential expected utility gain if Dan could get all the way up to 50% as seen here:

$$EU = \ln(W) + (1/2)\,[\ln(4*0.50+0.50) + \ln(0.50/4 + 0.50)$$

$$= \ln(W) + (1/2)\ln(2.5*0.625)$$

$$= \ln(W) + 0.223$$

The problem is that Dan can't borrow to leverage his investments at 10:1. Going to 2:1 is a step in the right direction. That is all we are saying.